Nucleon and cluster emission in electron scattering

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Abstract. The electrodisintegration of nuclei into two fragments is investigated, and formulae are derived for the cross section. It is assumed that the nucleus in its initial state is made up of the two clusters interacting with each other to form a bound state. The theory is applied to the problem of single-nucleon emission in order to investigate the importance of indirect processes in the quasi-elastic peak. These are found to reduce the cross section by a few per cent, which is not enough to give agreement with experiment.

1. Introduction

Electrodisintegration is becoming increasingly important in the study of nuclear structure. The theory of single-nucleon emission has been considered in some detail (Czyż 1963, Czyż and Gottfried 1963, de Forest 1967, de Forest and Walecka 1966, Murray and Strachan 1965, Strachan and Watt 1969, to be referred to as I), but agreement with experiment is not yet very good. Czyż managed to obtain reasonable agreement with experiment, but only by using a Fermi gas model for ^{12}C, which, as he points out, is rather unsatisfactory. Also the Fermi gas model failed to explain the quasi-elastic scattering from ^{208}Bi (Isabelle and Kendall 1964), where one would expect it to work well. In I, the shape of the quasi-elastic spectrum could be fitted by using only the direct electron–nucleon interaction, but the calculated magnitude of the cross section was about twice the experimental magnitude. From sum-rule considerations (McVoy and Van Hove 1962) we believe that indirect processes may interfere with the direct reaction to reduce the cross section.

There has also been some interest recently in processes in which clusters of nucleons, for example deuterons, are ejected from nuclei by the scattering of electrons (Griffy *et al.* 1966). However, the theory of cluster emission requires some clarification. Indirect effects will be even more important in cluster emission than in single-nucleon emission, since the relative importance of direct and indirect effects is determined by the ratio of the masses of the fragments. If, as suggested above, the indirect terms are appreciable in single-nucleon emission, then their neglect in the emission of larger fragments could be serious.

In this paper our object is to give a better description of the quasi-elastic peak in electron scattering. This is largely due to the direct knock-out of a single nucleon, but we wish to consider quantitatively the effects of including indirect reactions. Since nucleon emission is a special case of cluster emission, methods will be developed to describe disintegration into two arbitrary fragments. These methods will finally be applied to the problem of single-nucleon emission. As in I, we consider the quasi-elastic peak in ^{12}C.

2. Kinematics and nuclear model

Suppose that the nucleus disintegrates into two fragments of A and B nucleons. They will be referred to as A and B respectively. Let M_A, M_B, $\boldsymbol{R}_A$, $\boldsymbol{R}_B$ be the masses and centre-of-mass coordinates of A and B. Then we can write

$$M = M_A + M_B$$
$$\boldsymbol{r} = \boldsymbol{R}_A - \boldsymbol{R}_B \tag{1}$$
$$\boldsymbol{R} = \frac{1}{M}(M_A\boldsymbol{R}_A + M_B\boldsymbol{R}_B).$$

Let the coordinate variable for nucleon α of A be $\boldsymbol{r}_\alpha$, $\alpha \in A$, and for nucleon β of B be $\boldsymbol{r}_\beta$,

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$\beta \in B$, and let

$$x_\alpha = r_\alpha - R_A$$
$$x_\beta = r_\beta - R_B. \tag{2}$$

Then x_α, x_β are coordinates measured relative to the centre of mass of A, B and

$$\sum_{\alpha \in A} x_\alpha = \sum_{\beta \in B} x_\beta = 0. \tag{3}$$

Let p_α, P_A, p, P and π_α be momentum operators conjugate to r_α, R_A, r, R and x_α, respectively, and let μ be the reduced mass:

$$\mu = \frac{M_A M_B}{M_A + M_B} = \frac{M_A M_B}{M}. \tag{4}$$

Then

$$P = P_A + P_B$$
$$\pi_\alpha = \left(p_\alpha - \frac{1}{A} P_A\right) \frac{A}{A-1}$$

and

$$\tag{5}$$

$$p = \mu\left(\frac{P_A}{M_A} - \frac{P_B}{M_B}\right).$$

The kinetic energy operator may be transformed into

$$T = \frac{P^2}{2M} + \frac{p^2}{2\mu} + T_A + T_B. \tag{6}$$

If it is assumed that the nucleons interact through a two-body potential $V(ij)$, the total potential energy may be split up as follows:

$$V = \tfrac{1}{2} \sum_{i \neq j} V(ij) = \tfrac{1}{2} \sum_{\substack{i,j \in A \\ i \neq j}} V(ij) + \tfrac{1}{2} \sum_{\substack{i,j \in B \\ i \neq j}} V(ij) + \sum_{\substack{i \in A \\ j \in B}} V(ij) \tag{7}$$

$$= V_A + V_B + V(r) + H_{int}. \tag{8}$$

We assume that the interaction between A and B can be represented by a convenient central potential $V(r)$ and a residual part H_{int}. The Hamiltonian for the nuclear system may then be written

$$H = T + V = \frac{P^2}{2M} + H_0 + H_A + H_B + H_{int} \tag{9}$$

where

$$H_0 = \frac{p^2}{2\mu} + V(r), \qquad H_{A,B} = T_{A,B} + V_{A,B}. \tag{10}$$

We assume that any state of the nuclear system, whether before or after disintegration, may be written as a linear combination of the basic states

$$\exp(i\mathbf{k}.\mathbf{R})|\gamma\rangle|\alpha\rangle|\beta\rangle \tag{11}$$

where $|\gamma\rangle$, $|\alpha\rangle$ and $|\beta\rangle$ are eigenstates of H_0, H_A and H_B. We shall suppose that H_{int} may be neglected in the final state, so that we may describe the system after disintegration by the single ket

$$\exp(i\mathbf{q}.\mathbf{R})|\gamma_f\rangle|\alpha_f\rangle|\beta_f\rangle. \tag{12}$$

Before disintegration we describe the system by the linear combination

$$\sum_{\alpha\beta\gamma} c_{\alpha\beta\gamma}|\gamma\rangle|\alpha\rangle|\beta\rangle \tag{13}$$

where we have supposed the initial nucleus to be at rest in the laboratory. The coefficients $c_{\alpha\beta\gamma}$ could be obtained by diagonalizing H in the space spanned by kets like (11).

3. The interaction Hamiltonian

The Hamiltonian describing the interaction between electron and nucleus is proportional to (McVoy and Van Hove 1962)†

$$H' = \left(\sum_{j\in A} + \sum_{j\in B}\right)\left[\left\{e_j + \frac{q_\mu{}^2}{8m^2}(e_j - 2\mu_j)\right\}\exp(i\boldsymbol{q}.\boldsymbol{r}_j) - \boldsymbol{\alpha}.\frac{e_j}{2m}\left\{\boldsymbol{p}_j\exp(i\boldsymbol{q}.\boldsymbol{r}_j) + \exp(i\boldsymbol{q}.\boldsymbol{r}_j)\boldsymbol{p}_j\right\}\right.$$

$$\left. + i\boldsymbol{\alpha}.\frac{\mu_j}{2m}(\boldsymbol{q}\times\boldsymbol{\sigma}_j)\exp(i\boldsymbol{q}.\boldsymbol{r}_j)\right]. \tag{14}$$

Here e_j, $\mu_j\boldsymbol{\sigma}_j$ and m are the charge, magnetic moment, spin and mass of the jth nucleon, α_x, α_y, α_z are Dirac matrices for the electron, and $q_\mu{}^2 = q^2 - \omega^2$, where $\boldsymbol{q}$ and ω are the momentum and energy transferred from the electron, respectively. Taking matrix elements between plane-wave eigenstates of $\boldsymbol{P}$ gives a momentum δ-function, which we omit, and replaces $\boldsymbol{P}$ by 0 if the nucleus is at rest in the laboratory. Writing

$$H_A' = \sum_{j\in A}\left[\left\{e_j + \frac{q_\mu{}^2}{8m^2}(e_j - 2\mu_j)\right\}\exp(i\boldsymbol{q}.\boldsymbol{x}_j) - \boldsymbol{\alpha}.\frac{e_j}{2m}\left\{\boldsymbol{\pi}_j\exp(i\boldsymbol{q}.\boldsymbol{x}_j) + \exp(i\boldsymbol{q}.\boldsymbol{x}_j)\boldsymbol{\pi}_j\right\}\left(1 - \frac{1}{A}\right)\right.$$

$$\left. + \boldsymbol{\alpha}.\frac{i\mu_j}{2m}\left\{\boldsymbol{q}\times\boldsymbol{\sigma}_j\exp(i\boldsymbol{q}.\boldsymbol{x}_j)\right\}\right] \tag{15}$$

and a similar expression for H_B', we find

$$H' = \exp\left(i\boldsymbol{q}.\boldsymbol{r}\frac{\mu}{M_A}\right)H_A' + \exp\left(-i\boldsymbol{q}.\boldsymbol{r}\frac{\mu}{M_B}\right)H_B'$$

$$-\left\{\sum_{\alpha\in A}e_\alpha\exp(i\boldsymbol{q}.\boldsymbol{x}_\alpha)\right\}\left[\frac{1}{2M_A}\boldsymbol{\alpha}.\left\{\boldsymbol{p}\exp\left(i\boldsymbol{q}.\boldsymbol{r}\frac{\mu}{M_A}\right) + \exp\left(i\boldsymbol{q}.\boldsymbol{r}\frac{\mu}{M_A}\right)\boldsymbol{p}\right\}\right.$$

$$\left. + \frac{1}{2M}\boldsymbol{\alpha}.\boldsymbol{q}\exp\left(i\boldsymbol{q}.\boldsymbol{r}\frac{\mu}{M_A}\right)\right]$$

$$-\left\{\sum_{\beta\in B}e_\beta\exp(i\boldsymbol{q}.\boldsymbol{x}_\beta)\right\}\left[\frac{1}{2M_B}\boldsymbol{\alpha}.\left\{-\boldsymbol{p}\exp\left(-i\boldsymbol{q}.\boldsymbol{r}\frac{\mu}{M_B}\right) - \exp\left(-i\boldsymbol{q}.\boldsymbol{r}\frac{\mu}{M_B}\right)\boldsymbol{p}\right\}\right.$$

$$\left. + \frac{1}{2M}\boldsymbol{\alpha}.\boldsymbol{q}\exp\left(-i\boldsymbol{q}.\boldsymbol{r}\frac{\mu}{M_B}\right)\right]. \tag{16}$$

The expression for H_A' is the operator one would use to describe the interaction of an electron with a 'nucleus' of A particles, after transformation with the Gartenhaus–Schwartz transformation (McVoy and Van Hove 1962).

Equation (16) is of interest in situations other than the present one, and can be used in a suitably modified form in models in which the nucleus is supposed to be made up from two different types of particles, as in the Goldhaber–Teller (1948) model, or in more complicated situations (Raphael *et al.* 1966). It is of interest to note that this Hamiltonian can cause disintegration into two fragments together with a change in state of these fragments. In particular, it allows emission of more than one nucleon.

The usual procedure for expanding in multipoles (de Forest and Walecka 1966) cannot now be carried out because the operator is not a sum of single-particle operators. However, it is possible to adopt a procedure similar to that used before. We start off by expressing the various terms in H' as sums of tensor operators in standard fashion; this gives an expression containing products of pairs of tensor operators. These are then combined into

† We use natural units $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$.

single tensor operators, and the resulting expression resembles the simpler form used in the direct interaction picture. The Wigner–Eckart theorem (Edmonds 1957) may then be used on the matrix elements of the operator.

4. Expansion in multipoles

We start off by finding an expansion for H_A' of equation (15). As mentioned above, H_A' is just the operator describing the scattering of an electron from a 'nucleus' A, which has been considered in considerable detail in the past (de Forest and Walecka 1966), and so the problem would appear to be solved. However, H_A' is related to the operator usually employed by the Gartenhaus–Schwartz transformation (McVoy and Van Hove 1962), and the form (15) is hardly ever used in discussing scattering processes in which the nucleus is left in its ground state or in a low-lying excited state. What is done (de Forest and Walecka 1966) is to use the untransformed operator (equation (14), summed only over $j \in A$) and to make an adjustment to the matrix elements to allow for the motion of the centre of mass (Tassie and Barker 1958). This procedure is well established for the harmonic oscillator model, and is very convenient to use, since it avoids the A-body operators contained in (15). The usual expansion in multipoles (Willey 1963, de Forest and Walecka 1966) is still valid for H_A' of (15), but in practice it may be more convenient to revert to (14) and make the adjustment for the motion of the centre of mass at the end of the calculation.

We prefer to expand (15) as it stands, and obtain

$$H_A' = \sum_{lm} [Y_{lm}^*(\hat{q})\tilde{T}_{lm}^{(1)}(A) + \alpha.q\, Y_{lm}^*(\hat{q})T_{lm}^{(2)}(A) + \alpha.Y_{llm}^*(\hat{q})T_{lm}^{(3)}(A)$$
$$+ \alpha.\{q \times Y_{llm}^*(\hat{q})\}T_{lm}^{(4)}(A)] \tag{17}$$

where $\hat{q}$ is a unit vector in the direction of q and Y_{llm} is a vector spherical harmonic (Edmonds 1957). Explicit forms for the tensor operators T_{lm} may be obtained from Willey (1963) or from de Forest and Walecka (1966). For example,

$$\tilde{T}_{lm}^{(1)}(A) = 4\pi \sum_{\alpha \in A} i^l \left\{ e_\alpha + \frac{q_\mu^2}{8m^2}(e_\alpha - 2\mu_\alpha) \right\} j_l(qx_\alpha) Y_{lm}(\hat{x}_\alpha). \tag{18}$$

Returning to equation (16), we next write

$$\exp\left(iq.r\frac{\mu}{M_A}\right) = \sum_{LM} Y_{LM}^*(\hat{q})T_{LM}^{(1)}(a) \tag{19}$$

where

$$T_{LM}^{(1)}(a) = 4\pi i^L j_L\left(qr\frac{\mu}{M_A}\right) Y_{LM}(\hat{r}). \tag{20}$$

Finally, the remaining parts of equation (16) are written

$$-\frac{1}{2M_A}\alpha.\left\{p\exp\left(iq.r\frac{\mu}{M_A}\right) + \exp\left(iq.r\frac{\mu}{M_A}\right)p\right\} - \frac{1}{2M}\alpha.q\exp\left(iq.r\frac{\mu}{M_A}\right)$$

$$= \sum_{LM}\left[\alpha.q\, Y_{LM}^*(\hat{q})\left\{T_{LM}^{(2)}(a) - \frac{1}{2M}T_{LM}^{(1)}(a)\right\} + \alpha.Y_{LLM}^*(\hat{q})T_{LM}^{(3)}(a)\right.$$

$$\left. + \alpha.\{q \times Y_{LLM}^*(\hat{q})\}T_{LM}^{(4)}(a)\right] \tag{21}$$

and

$$\sum_{j \in A} e_j \exp(iq.x_j) = \sum_{lm} Y_{lm}^*(\hat{q})\mathcal{T}_{lm}^{(1)}(A). \tag{22}$$

Tensors operating on internal coordinates of A are labelled $T(A)$, while those operating on the relative coordinate r but containing A in a minor way, for example as in (20), are written $T(a)$. Operators $\tilde{T}_{\lambda\mu}^{(1)}(B), T_{\lambda\mu}^{(1)}(b)$ etc. are defined in a similar way.

Using the above expressions, we may rewrite equation (16) as follows:

$$H' = \sum_{LMlm} \left(\left\{ Y^*_{LM}(\hat{q}) T^{(1)}_{LM}(a) \right\} \left\{ Y^*_{lm} \tilde{T}^{(1)}_{lm}(A) + \alpha.q\, Y^*_{lm} T^{(2)}_{lm}(A) + \alpha.Y^*_{llm} T^{(3)}_{lm}(A) \right. \right.$$

$$\left. + \alpha.q \times Y^*_{llm} T^{(4)}_{lm}(A) \right\}$$

$$+ \left\{ Y^*_{LM} \mathscr{T}^{(1)}_{LM}(A) \right\} \left[\alpha.q\, Y^*_{lm} \left\{ T^{(2)}_{lm}(a) - \frac{1}{2M} T^{(1)}_{lm}(a) \right\} + \alpha.Y^*_{llm} T^{(3)}_{lm}(a) \right.$$

$$\left. + \alpha.q \times Y^*_{llm} T^{(4)}_{lm}(a) \right]$$

$$\left. + \text{a similar expression with A, a replaced by B, b} \right). \tag{23}$$

Rearranging terms, we have

$$H' = \sum_{LMlm} Y^*_{LM}(\hat{q}) \left(Y^*_{lm}(\hat{q}) [T^{(1)}_{LM}(a) \tilde{T}^{(1)}_{lm}(A) + T^{(1)}_{LM}(b) \tilde{T}^{(1)}_{lm}(B)] \right.$$

$$+ \alpha.q\, Y^*_{lm}(\hat{q}) [T^{(1)}_{LM}(a) T^{(2)}_{lm}(A) + T^{(1)}_{LM}(b) T^{(2)}_{lm}(B) + \mathscr{T}^{(1)}_{LM}(A) T^{(2)}_{lm}(a)$$

$$+ \mathscr{T}^{(1)}_{LM}(B) T^{(2)}_{lm}(b) - \frac{1}{2M} \{ \mathscr{T}^{(1)}_{LM}(A) T^{(1)}_{lm}(a) + \mathscr{T}^{(1)}_{LM}(B) T^{(1)}_{lm}(b) \}]$$

$$+ \alpha.Y^*_{llm}(\hat{q}) [T^{(1)}_{LM}(a) T^{(3)}_{lm}(A) + T^{(1)}_{LM}(b) T^{(3)}_{lm}(B) + \mathscr{T}^{(1)}_{LM}(A) T^{(3)}_{lm}(a)$$

$$+ \mathscr{T}^{(1)}_{LM}(B) T^{(3)}_{lm}(b)] + \alpha.\{q \times Y^*_{llm}(\hat{q})\} [T^{(1)}_{LM}(a) T^{(4)}_{lm}(A) + T^{(1)}_{LM}(b) T^{(4)}_{lm}(B)$$

$$\left. + \mathscr{T}^{(1)}_{LM}(A) T^{(4)}_{lm}(a) + \mathscr{T}^{(1)}_{LM}(B) T^{(4)}_{lm}(b)] \right). \tag{24}$$

Defining new tensor operators $T^{(j)}_{\lambda\mu}$ depending on L, l but not on M, m by

$$T^{(j)}_{\lambda\mu}(Ll) = \sum_{M, m} (LMlm|Ll\lambda\mu) [T^{(1)}_{LM}(a) T^{(j)}_{lm}(A) + \dots] \tag{25}$$

where the terms in square brackets are written in full in equation (24), we obtain, since

$$[T^{(1)}_{LM}(a) T^{(j)}_{lm}(A) + \dots] = \sum_{\lambda\mu} T^{(j)}_{\lambda\mu}(Ll)\hat{\lambda}(-)^{L-l+\mu} \begin{pmatrix} L & l & \lambda \\ M & m & -\mu \end{pmatrix} \tag{26}$$

$$H' = \sum_{\substack{LMlm \\ \lambda\mu}} (-)^{L-l+\mu}\hat{\lambda} \begin{pmatrix} L & l & \lambda \\ M & m & -\mu \end{pmatrix} \{ Y^*_{LM} Y^*_{lm} \tilde{T}^{(1)}_{\lambda\mu}(Ll)$$
$$+ \alpha.q\, Y^*_{LM} Y^*_{lm} T^{(2)}_{\lambda\mu}(Ll) + \alpha.\, Y^*_{LM} Y^*_{llm} T^{(3)}_{\lambda\mu}(Ll)$$
$$+ \alpha.q \times Y^*_{LM} Y^*_{llm} T^{(4)}_{\lambda\mu}(Ll) \}. \tag{27}$$

$(LMlm|Ll\lambda\mu)$ is a Clebsch–Gordan coefficient and $\begin{pmatrix} L & l & \lambda \\ M & m & -\mu \end{pmatrix}$ is a 3–j symbol as defined by Edmonds (1957). Also $\hat{\lambda} = (2\lambda+1)^{1/2}$.

The summations over M, m may be performed as follows, where we use several results given by Edmonds (1957):

$$\sum_{Mm} (-)^{\mu} \begin{pmatrix} L & l & \lambda \\ M & m & -\mu \end{pmatrix} Y^*_{LM} Y^*_{lm} = \frac{\hat{L}\hat{l}}{\hat{\lambda}(4\pi)^{1/2}} \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} Y^*_{\lambda\mu}.$$

Using the 6–j symbol $\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$ as defined by Edmonds (1957), the other result is

$$\sum_{Mm} (-)^\mu \begin{pmatrix} L & l & \lambda \\ M & m & -\mu \end{pmatrix} Y^*_{LM} Y^*_{llm}$$

$$= \sum_{\lambda'} \begin{Bmatrix} \lambda' & 1 & \lambda \\ l & L & l \end{Bmatrix} \begin{pmatrix} L & l & \lambda' \\ 0 & 0 & 0 \end{pmatrix} (-)^{\lambda+\lambda'+1} Y^*_{\lambda\lambda'\mu} \frac{(2l+1)\hat{L}\hat{\lambda}'}{\hat{\lambda}(4\pi)^{1/2}}.$$

Hence expression (27) becomes

$$H' = \sum_{\substack{Ll \\ \lambda\mu}} \frac{\hat{L}\hat{l}}{(4\pi)^{1/2}} \Bigg[(-)^\lambda \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} Y^*_{\lambda\mu} \{ \tilde{T}^{(1)}_{\lambda\mu}(Ll) + \boldsymbol{\alpha}.\boldsymbol{q} T^{(2)}_{\lambda\mu}(Ll)\}$$

$$+ \boldsymbol{\alpha}.\sum_{\lambda'} \begin{Bmatrix} \lambda' & 1 & \lambda \\ l & L & l \end{Bmatrix} \begin{pmatrix} L & l & \lambda' \\ 0 & 0 & 0 \end{pmatrix} (-)^{\lambda+1} \hat{l}\hat{\lambda}' \{ T^{(3)}_{\lambda\mu}(Ll) Y^*_{\lambda\lambda'\mu}$$

$$+ T^{(4)}_{\lambda\mu}(Ll) \boldsymbol{q} \times Y^*_{\lambda\lambda'\mu} \} \Bigg]. \tag{28}$$

Since

$$\boldsymbol{q}.Y^*_{llm} = 0 = \boldsymbol{q}.(\boldsymbol{q} \times Y_{llm})$$

it is clear from (27) that the vector

$$\sum_{\lambda'} \begin{Bmatrix} \lambda' & 1 & \lambda \\ l & L & l \end{Bmatrix} \begin{pmatrix} L & l & \lambda' \\ 0 & 0 & 0 \end{pmatrix} (-)^{\lambda+1} \hat{l}\hat{\lambda}' \{ T^{(3)}_{\lambda\mu}(Ll) Y^*_{\lambda\lambda'\mu} + T^{(4)}_{\lambda\mu}(Ll) \boldsymbol{q} \times Y^*_{\lambda\lambda'\mu} \}$$

is perpendicular to $\boldsymbol{q}$. Hence we expect the second part of (28), $\boldsymbol{\alpha}.\Sigma_{\lambda'}...$ to give eventually a transverse form factor, while the first part of (28), which contains no transverse terms, will give a longitudinal form factor.

We shall consider in detail only the part of (28) which contains $\tilde{T}^{(1)}_{\lambda\mu}(Ll)$, and this we shall call $\mathcal{O}$. If we take matrix elements of $\mathcal{O}$ between nuclear states of initial and final angular momentum J_i, J_f with z components M_i, M_f respectively, and electron spin states $|S_i\rangle$, $|S_f\rangle$, we find

$$(S_f J_f M_f | \mathcal{O} | S_i J_i M_i) = \sum_{Ll\lambda\mu} (-)^{L-l} \frac{\hat{L}\hat{l}}{(4\pi)^{1/2}} \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} Y^*_{\lambda\mu}(S_f|S_i)$$

$$\times (J_f||\tilde{\boldsymbol{T}}_\lambda^{(1)}(Ll)||J_i)(-)^{J_f-M_f} \begin{pmatrix} J_f & \lambda & J_i \\ -M_f & \mu & M_i \end{pmatrix}. \tag{29}$$

$(J_f||\hat{\boldsymbol{T}}_\lambda||J_i)$ is a reduced matrix element, and the Wigner–Eckart theorem has been used. We now sum the square of the modulus of this matrix element over M_i, M_f and we obtain

$$\sum_{M_f M_i} |(S_f J_f M_f | \mathcal{O} | S_i J_i M_i)|^2 = \sum_{\substack{LlL'l' \\ \lambda\mu}} (-)^{L-l+L'-l'} \frac{\hat{L}\hat{l}\hat{L}'\hat{l}'}{4\pi} \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times |(S_f|S_i)|^2 \frac{|Y_{\lambda\mu}|^2}{2\lambda+1} (J_f||\tilde{\boldsymbol{T}}_\lambda^{(1)}(Ll)||J_i)(J_f||\tilde{\boldsymbol{T}}_\lambda^{(1)}(L'l')||J_i)^*. \tag{30}$$

It is now possible to sum over the electron spins using the usual trace methods, and also to

sum over μ. We find

$$\sum_{\substack{M_i M_f \\ S_i S_f}} |(S_f J_f M_f|\mathcal{O}|S_i J_i M_i)|^2 = \sum_{L l L' l' \lambda} (1+\cos\theta) \frac{\hat{L}\hat{l}\hat{L}'\hat{l}'}{(4\pi)^2} \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times (J_f||\tilde{\boldsymbol{T}}_\lambda^{(1)}(Ll)||J_i)(J_f||\tilde{\boldsymbol{T}}_\lambda^{(1)}(L'l')||J_i)^* \qquad (31)$$

where θ is the angle through which the electron is scattered.

The remainder of the operator H' of equation (28) may be dealt with in a similar way, but the algebra is more difficult than that given above. In addition to the above sum over μ, others of a similar type are required. The results are given in the appendix.

Writing

$$T_\lambda(Ll) = (J_f||\boldsymbol{T}_\lambda(Ll)||J_i) \qquad (32)$$

we obtain finally

$$\sum_{\substack{M_f M_i \\ S_f S_i}} |(S_f J_f M_f|H'|S_i J_i M_i)|^2 = 2\cos^2{\tfrac{1}{2}\theta} \sum_{\lambda L L' l l'} \frac{\hat{L}\hat{L}'\hat{l}\hat{l}'}{(4\pi)^2} \left\{ \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \{\tilde{T}_\lambda^{(1)}(Ll) \right.$$

$$+ \omega T_\lambda^{(2)}(Ll)\}\{\tilde{T}_\lambda^{(1)}(L'l') + \omega T_\lambda^{(2)}(L'l')\}^*$$

$$+ \sum_{\lambda' \lambda''} \hat{l}\hat{l}'\hat{\lambda}'\hat{\lambda}'' \begin{Bmatrix} \lambda' & 1 & \lambda \\ l & L & l \end{Bmatrix} \begin{Bmatrix} \lambda'' & 1 & \lambda \\ l' & L' & l' \end{Bmatrix} \begin{pmatrix} L & l & \lambda' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & l' & \lambda'' \\ 0 & 0 & 0 \end{pmatrix}$$

$$\times \left(T_\lambda^{(3)}(Ll)T_\lambda^{(3)}(L'l')^* \left[\frac{2}{3}\left(\frac{1}{2} + \tan^2{\tfrac{1}{2}\theta}\right)\delta_{\lambda'\lambda''} \right. \right.$$

$$+ \left\{ \frac{2}{3}\left(\frac{1}{2} + \tan^2{\tfrac{1}{2}\theta}\right) - \left(\frac{q_\mu^2}{2q^2} + \tan^2{\tfrac{1}{2}\theta}\right) \right\}$$

$$\left. \times (-)^\lambda \sqrt{30} \begin{Bmatrix} 2 & 1 & 1 \\ \lambda & \lambda' & \lambda'' \end{Bmatrix} \begin{pmatrix} \lambda' & \lambda'' & 2 \\ 0 & 0 & 0 \end{pmatrix} \hat{\lambda}'\hat{\lambda}'' \right]$$

$$+ T_\lambda^{(4)}(Ll)T_\lambda^{(4)}(L'l')^* \left(\frac{q_\mu^2}{2q^2} + \tan^2{\tfrac{1}{2}\theta}\right)$$

$$\left. \left. \times \frac{2}{3}\left[\delta_{\lambda'\lambda''} + (-)^{\lambda+1} \sqrt{\frac{15}{2}} \begin{Bmatrix} 2 & 1 & 1 \\ \lambda & \lambda' & \lambda'' \end{Bmatrix} \begin{pmatrix} \lambda' & \lambda'' & 2 \\ 0 & 0 & 0 \end{pmatrix} \hat{\lambda}'\hat{\lambda}'' \right] \right) \right\}. \qquad (33)$$

In order to eliminate terms of the type $T_\lambda^{(3)*}T_\lambda^{(4)}$ it is necessary to use the relations

$$\tilde{T}_\lambda^{(1)}(Ll)^* = (-)^{J_i - J_f + L + l}\tilde{T}_\lambda^{(1)}(Ll)$$

$$T_\lambda^{(2)}(Ll)^* = (-)^{J_i - J_f + L + l}T_\lambda^{(2)}(Ll)$$

$$T_\lambda^{(3)}(Ll)^* = (-)^{J_i - J_f + L + l - 1}T_\lambda^{(3)}(Ll) \qquad (34)$$

$$T_\lambda^{(4)}(Ll)^* = (-)^{J_i - J_f + L + l + 1}T_\lambda^{(4)}(Ll).$$

These may be easily derived from the definitions (25), and the known properties of the operators $T_{LM}^{(1)}(A)$ etc. (de Forest and Walecka 1966).

Expression (33) has the form

$$\cos^2{\tfrac{1}{2}\theta}(A + B\tan^2{\tfrac{1}{2}\theta}) \qquad (35)$$

as it must, since it is appropriate for an experiment in which only the scattered electron is detected (de Forest and Walecka 1966, equation 2.13).

It is possible to combine the sum $\tilde{T}_{\lambda\mu}^{(1)}(Ll) + \omega T_{\lambda\mu}^{(2)}(Ll)$ into a single operator, so that we may write

$$
\begin{aligned}
T_{\lambda\mu}^{(1)}(Ll) &= \tilde{T}_{\lambda\mu}^{(1)}(Ll) + \omega T_{\lambda\mu}^{(2)}(Ll) \\
&= \sum_{Mm} (LMlm|Ll\lambda\mu)\{T_{LM}^{(1)}(a)T_{lm}^{(1)}(A) + T_{LM}^{(1)}(b)T_{lm}^{(1)}(B)\} \\
&\equiv \{T_L^{(1)}(a) \otimes T_l^{(1)}(A)\}_{\lambda\mu} + \{T_L^{(1)}(b) \otimes T_l^{(1)}(B)\}_{\lambda\mu}
\end{aligned}
\tag{36}
$$

where

$$
\begin{aligned}
T_{LM}^{(1)}(A) &= \tilde{T}_{LM}^{(1)}(A) - \frac{\omega^2}{q^2}\mathscr{T}_{LM}^{(1)}(A) \\
&= \frac{q_\mu^2}{q^2} \sum_{j\in A}\left\{e_j + \frac{q^2}{8m^2}(e_j - 2\mu_j)\right\}4\pi i^L j_L(qx_j)Y_{LM}(\hat{\boldsymbol{x}}_j).
\end{aligned}
\tag{37}
$$

5. Summations over final-state quantum numbers

The next step is to express the reduced matrix elements $T_\lambda(Ll)$ as products of reduced matrix elements of operators operating on the various parts of the nuclear system, and then to sum over the unobserved quantum numbers.

Before proceeding it is necessary to have a definite scheme for coupling the individual angular momenta to J_i and J_f, since the phases of the reduced matrix elements depend on the order of coupling.

We use $\{J_A, \dot{J}_A, ...\}$ to denote possible values of the angular momentum of A in the initial state. These are coupled to the set $\{J_B, \dot{J}_B, ...\}$ for B to form $\{\alpha, \dot{\alpha}, ...\}$, thus:

$$
\begin{aligned}
\boldsymbol{\alpha} &= \boldsymbol{J}_A + \boldsymbol{J}_B \\
\dot{\boldsymbol{\alpha}} &= \dot{\boldsymbol{J}}_A + \dot{\boldsymbol{J}}_B, \qquad \boldsymbol{\alpha}_1 = \dot{\boldsymbol{J}}_A + \boldsymbol{J}_B, \dots .
\end{aligned}
\tag{39}
$$

The set $\{\alpha, \dot{\alpha}, ...\}$ are then coupled to the possible values of the relative angular momentum $\{J, \dot{J}, ...\}$ to form J_i according to

$$
\begin{aligned}
\boldsymbol{J}_i &= \boldsymbol{\alpha} + \boldsymbol{J} \\
\boldsymbol{J}_i &= \dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{J}}, \dots .
\end{aligned}
\tag{40}
$$

In the final state we suppose that the clusters have definite angular momenta J_A', J_B' and that they are coupled to α', which is then coupled to the relative part J' to give J_f. That is,

$$
\boldsymbol{\alpha}' = \boldsymbol{J}_A' + \boldsymbol{J}_B', \qquad \boldsymbol{J}_f = \boldsymbol{\alpha}' + \boldsymbol{J}'.
\tag{41}
$$

Unobserved final-state quantum numbers are J_f, α', J', and it is therefore necessary to sum over these.

A typical basic state of the system can be represented by the ket $|\gamma_A J_A \gamma_B J_B \alpha; \gamma J; J_i\rangle$, and the initial state of the nucleus may be represented by a linear combination of such kets. The final state may be represented by a single such ket provided that we sum over the unobserved quantum numbers given above. At present, the additional quantum numbers γ_A, γ_B, γ will be ignored, and we consider for simplicity only one ket in the initial state.

From here on only the first term of equation (33), summed over J_f, α', J', will be considered, and we write it as

$$
\frac{2\cos^2\tfrac{1}{2}\theta}{(4\pi)^2} \sum_{LL'll'J'} \hat{L}\hat{L}'\hat{l}\hat{l}'E
$$

where

$$
E = \sum_{\lambda J_f \alpha'} \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} L' & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix} T_\lambda^{(1)}(Ll)T_\lambda^{(1)*}(L'l').
\tag{42}
$$

The reduced matrix element $T_\lambda^{(1)}(L, l)$ becomes (Edmonds 1957)

$$T_\lambda^{(1)}(Ll) \sim (J_A{}'J_B{}'\alpha'J'J_f||[\{T_L^{(1)}(a) \otimes T_l^{(1)}(A)\}_\lambda + \{T_L^{(1)}(b) \otimes T_l^{(1)}(B)\}_\lambda]||J_AJ_B\alpha JJ_i)$$

$$= (J_A{}'J_B{}'\alpha'J'J_f||[\{T_l^{(1)}(A) \otimes T_L^{(1)}(a)\}_\lambda + \{T_l^{(1)}(B) \otimes T_L^{(1)}(b)\}_\lambda]||J_AJ_B\alpha JJ_i)$$

$$= \hat{J}_f\hat{J}_i\hat{\lambda} \begin{Bmatrix} \alpha' & \alpha & l \\ J' & J & L \\ J_f & J_i & \lambda \end{Bmatrix} [(J_A{}'J_B{}'\alpha'||T_l^{(1)}(A)||J_AJ_B\alpha)(J'||T_L^{(1)}(a)||J)$$

$$+ (J_A{}'J_B{}'\alpha'||T_l^{(1)}(B)||J_AJ_B\alpha)(J'||T_L^{(1)}(b)||J)]$$

$$= \hat{J}_f\hat{J}_i\hat{\lambda}\hat{\alpha}\hat{\alpha}' \begin{Bmatrix} \alpha' & \alpha & l \\ J' & J & L \\ J_f & J_i & \lambda \end{Bmatrix} \left[(-)^{J_A{}'+J_B{}'+\alpha+l} \begin{Bmatrix} J_A{}' & \alpha' & J_B{}' \\ \alpha & J_A & l \end{Bmatrix} \right.$$

$$\times (J_A{}'||T_l^{(1)}(A)||J_A)(J'||T_L^{(1)}(a)||J)\delta_{J_B{}'J_B}$$

$$\left. + (-)^{J_A{}'+J_B+\alpha'+l} \begin{Bmatrix} J_B{}' & \alpha' & J_A{}' \\ \alpha & J_B & l \end{Bmatrix} (J_B{}'||T_l^{(1)}(B)||J_B)(J'||T_L^{(1)}(b)||J)\delta_{J_A{}'J_A} \right].$$

$$(43)$$

If we write

$$T_l^{(1)}(A; J_A{}'J_A) = (J_A{}'||T_l^{(1)}(A)||J_A) \tag{44}$$

and consider only the mutual interference between the components J_A, J_B, J, α and $\dot{J}_A$, $\dot{J}_B$, $\dot{J}$, $\dot{\alpha}$ of the initial state, we find from (42) and (43),

$$E = \sum_{\lambda J_i \alpha'} \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \hat{\alpha}\hat{\dot{\alpha}}(2\alpha'+1)\hat{J}_f{}^2\hat{J}_i{}^2\hat{\lambda}^2 \begin{Bmatrix} \alpha' & \alpha & l \\ J' & J & L \\ J_f & J_i & \lambda \end{Bmatrix} \begin{Bmatrix} \alpha' & \dot{\alpha} & l' \\ J' & \dot{J} & L' \\ J_f & J_i & \lambda \end{Bmatrix}$$

$$\times \left[(-)^{J_A{}'+J_B{}'+\alpha+l} \begin{Bmatrix} J_A{}' & \alpha' & J_B{}' \\ \alpha & J_A & l \end{Bmatrix} T_l^{(1)}(A; J_A{}'J_A)T_L^{(1)}(a; J'J)\delta_{J_B{}'J_B} \right.$$

$$\left. + (-)^{J_A{}'+J_B+\alpha'+l} \begin{Bmatrix} J_B{}' & \alpha' & J_A{}' \\ \alpha & J_B & l \end{Bmatrix} T_l^{(1)}(B; J_B{}'J_B)T_L^{(1)}(b; J'J)\delta_{J_A{}'J_A} \right]^*$$

$$\times \left(\text{a similar expression with } J_A \to \dot{J}_A, J_B \to \dot{J}_B, \alpha \to \dot{\alpha}, J \to \dot{J}\right). \tag{45}$$

Using the definition of the 9–j symbol as a sum of 6–j symbols, and the Elliott–Biedenharn identity (Edmonds 1957) the summations over J_f and λ may be performed, giving

$$\sum_{J_f \lambda} (2J_f+1)(2\lambda+1) \begin{pmatrix} L & l & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L' & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \alpha' & \alpha & l \\ J' & J & L \\ J_f & J_i & \lambda \end{Bmatrix} \begin{Bmatrix} \alpha' & \dot{\alpha} & l' \\ J' & \dot{J} & L' \\ J_f & J_i & \lambda \end{Bmatrix}$$

$$= \sum_k (-)^{\alpha'+J+\dot{J}+J'-J_i}(2k+1) \begin{Bmatrix} \dot{\alpha} & \alpha' & l' \\ l & k & \alpha \end{Bmatrix} \begin{Bmatrix} \dot{J} & J_i & \dot{\alpha} \\ \alpha & k & J \end{Bmatrix} \begin{Bmatrix} J & \dot{J} & k \\ L' & L & J' \end{Bmatrix} \begin{pmatrix} L & L' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & k \\ 0 & 0 & 0 \end{pmatrix}.$$

$$(46)$$

The sum over α' in equation (45) may now be carried out, and we finally obtain

$$E = \sum_k \begin{pmatrix} L & L' & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & k \\ 0 & 0 & 0 \end{pmatrix} \hat{\alpha}\hat{\dot{\alpha}}(2k+1)(2J_{\rm i}+1)(-)^{J+\dot{J}+J'-J_{\rm i}+k} \begin{Bmatrix} \dot{J} & J_{\rm i} & \dot{\alpha} \\ \alpha & k & J \end{Bmatrix} \begin{Bmatrix} J & \dot{J} & k \\ L' & L & J' \end{Bmatrix}$$

$$\times \left[T_l^{(1)}({\rm A};\, J_{\rm A}'J_{\rm A})^* T_L^{(1)}({\rm a};\, J'J) T_{l'}^{(1)}({\rm A};\, J_{\rm A}'\dot{J}_{\rm A}) T_{L'}^{(1)}({\rm a};\, J'\dot{J}) \delta_{J_{\rm B}'J_{\rm B}} \delta_{J_{\rm B}'\dot{j}_{\rm B}} \right.$$

$$\times (-)^{J_{\rm A}'+J_{\rm B}'-J_{\rm A}-\dot{j}_{\rm A}} \begin{Bmatrix} J_{\rm A} & \alpha & J_{\rm B}' \\ \dot{\alpha} & \dot{J}_{\rm A} & k \end{Bmatrix} \begin{Bmatrix} k & J_{\rm A} & \dot{J}_{\rm A} \\ J_{\rm A}' & l' & l \end{Bmatrix}$$

$$+ T_l^{(1)}({\rm B};\, J_{\rm B}'J_{\rm B})^* T_L^{(1)}({\rm b};\, J'J)^* T_{l'}^{(1)}({\rm B};\, J_{\rm B}'\dot{J}_{\rm B}) T_{L'}^{(1)}({\rm b};\, J'\dot{J}) \delta_{J_{\rm A}'J_{\rm A}} \delta_{J_{\rm A}'\dot{j}_{\rm A}}$$

$$\times (-)^{J_{\rm B}'-\alpha-\dot{\alpha}-J_{\rm A}'} \begin{Bmatrix} J_{\rm B} & \alpha & J_{\rm A}' \\ \dot{\alpha} & \dot{J}_{\rm B} & k \end{Bmatrix} \begin{Bmatrix} k & J_{\rm B} & \dot{J}_{\rm B} \\ J_{\rm B}' & l' & l \end{Bmatrix}$$

$$+ T_l^{(1)}({\rm A};\, J_{\rm A}'J_{\rm A})^* T_L^{(1)}({\rm a};\, J'J)^* T_{l'}^{(1)}({\rm B};\, J_{\rm B}'\dot{J}_{\rm B}) T_{L'}^{(1)}({\rm b};\, J'\dot{J}) \delta_{J_{\rm B}'J_{\rm B}} \delta_{J_{\rm A}'\dot{j}_{\rm A}}$$

$$\times (-)^{2J_{\rm A}'+J_{\rm B}'+\alpha+\dot{J}_{\rm B}} \begin{Bmatrix} \dot{\alpha} & J_{\rm A}' & \dot{J}_{\rm B} \\ \alpha & J_{\rm A} & J_{\rm B}' \\ k & l & l' \end{Bmatrix}$$

$$+ T_l^{(1)}({\rm B};\, J_{\rm B}'J_{\rm B})^* T_L^{(1)}({\rm b};\, J'J)^* T_{l'}^{(1)}({\rm A};\, J_{\rm A}'\dot{J}_{\rm A}) T_{L'}^{(1)}({\rm a};\, J'\dot{J}) \delta_{J_{\rm A}'J_{\rm A}} \delta_{J_{\rm B}'\dot{j}_{\rm B}}$$

$$\left. \times (-)^{2J_{\rm A}'+J_{\rm B}+J_{\rm B}'+\dot{\alpha}} \begin{Bmatrix} \dot{\alpha} & J_{\rm B}' & \dot{J}_{\rm A} \\ \alpha & J_{\rm B} & J_{\rm A}' \\ k & l & l' \end{Bmatrix} \right]. \tag{47}$$

Evaluation of expression (47) will in practice be far less difficult than it looks. Suppose, for example, that we were interested in the electrodisintegration of ^{6}Li into a ^{4}He nucleus and a deuteron. Letting A be ^{4}He and B be the deuteron, we have immediately

$$J_{\rm A}' = J_{\rm A} = \dot{J}_{\rm A} = 0, \qquad J_{\rm B}' = J_{\rm B} = \dot{J}_{\rm B} = 1, \qquad \alpha = \dot{\alpha} = 1$$

$$T_l^{(1)}({\rm A};\, J_{\rm A}'J_{\rm A}) = T_l^{(1)}({\rm A};\, 0\,0) = 0 \text{ unless } l = 0$$

$$T_l^{(1)}({\rm B};\, J_{\rm B}'J_{\rm B}) = T_l^{(1)}({\rm B};\, 1,\, 1) = 0 \text{ unless } l = 0,\, 2$$

and evaluation of the 6–j and 9–j symbols is easy.

If the final state of A labelled by $J_{\rm A}'$ does not contribute to the initial state of the nucleus,

$$\delta_{J_{\rm A}J_{\rm A}'} = \delta_{J_{\rm A}'\dot{j}_{\rm A}} = 0$$

and so only the first term in (47) can be non-zero, and then only if B is in the same state before and after the reaction. Thus at least one of the fragments does not change its state in the reaction.

The reduced matrix elements $T_l^{(1)}({\rm A};\, J_{\rm A}'J_{\rm A})$ are exactly the same as those which appear in the longitudinal form factor for electron scattering from a 'nucleus' A, when it makes the transition from initial state $J_{\rm A}$ to final state $J_{\rm A}'$, and similarly for $T_l^{(1)}({\rm B};\, J_{\rm B}'J_{\rm B})$. These matrix elements have been studied in detail (e.g. see de Forest and Walecka 1966).

Roughly speaking, the first term in equation (47) corresponds to scattering of the electron from A with simultaneous disintegration of the nucleus, while B is unaffected. The 'nucleus' A may be excited in this process, and may indeed itself disintegrate. In the second term, A is the spectator, while the scattering is from B. The third and fourth terms are interference terms, which only contribute if the final states of both A and B occur in the initial state.

Finally, if A is a single nucleon and we ignore all but the direct interaction term, which is the first term in (47), it is easy to verify that the expression reduces to give the longitudinal form factor already used in I.

When we return to equation (33), it is clear that the reduced matrix elements $T_\lambda^{(3)}(Ll)$ and $T_\lambda^{(4)}(Ll)$ can be split up to give expressions similar to (43), and that the summations over J_f, λ and α' may be performed as above. An expression resembling (47) may thus be obtained for the part of (33) which we have not considered in detail, but the above remarks will apply to it as well as to (47).

We shall now indicate briefly how the cross section is obtained from the above expressions. Denoting the set of quantum numbers $\{\gamma_A J_A \gamma_B J_B; \alpha; \gamma J\}$ by S, we may write the initial state as

$$|I; J_i\rangle = \sum_S A(S; I) |S; J_i\rangle.$$

Additional quantum numbers γ_A, γ_B, γ may be inserted in (47) by replacing $T_l^{(1)}(A; J_A' J_A)$ by $T_l^{(1)}(A; \gamma_A' J_A' \gamma_A J_A)$ etc., and by replacing $\delta_{J_A' J_A}$ by $\delta_{\gamma_A' \gamma_A} \delta_{J_A' J_A}$ etc. We denote this extended version of E by $E(S, \dot{S})$. Then the contribution to the cross section from the longitudinal part of the operator is, by the Fermi golden rule,

$$\frac{d^2\sigma}{dk_i d\Omega_f} = 2\pi \left\{ \left(\frac{4\pi\alpha^2}{q_\mu^2}\right)^2 f^2(q^2) \times \sum_{S\dot{S}} A^*(S; I) A(\dot{S}; I) \frac{2\cos^2 \frac{1}{2}\theta}{(4\pi)^2} \sum_{LL'll'} \hat{L}\hat{L}'\hat{l}\hat{l}' \frac{E(S\dot{S})}{2J_i+1} \right\}$$
$$\times \frac{1}{2} \frac{k_f^2}{(2\pi)^3} \rho_N(E_f). \tag{48}$$

The cross section is differential in the direction Ω_f and energy k_f of the scattered electron. The factor $4\pi\alpha/q_\mu^2$ converts the operator H' into the interaction Hamiltonian, $f(q^2)$ is the nucleon form factor, and the factor $2J_i+1$ comes from averaging over angular momentum states of the initial nucleus. The factor $\frac{1}{2}$ comes from averaging over the spin states of the incident electron beam, and the $k_f^2/(2\pi)^3$ from the density of states of the scattered electrons. The factor $\rho_N(E_f)$ is the density of final states of the nuclear system.

The Mott cross section for the scattering of an electron from a point charge is

$$\sigma_M = \frac{4\alpha^2 k_f^2 \cos^2 \frac{1}{2}\theta}{q_\mu^4}$$

and so we may write

$$\frac{d^2\sigma}{dk_f d\Omega_f} = \sigma_M |F_L|^2$$

where

$$|F_L|^2 = \left\{ \sum_{\substack{S\dot{S} \\ LL'll'}} A^*(S; I) A(\dot{S}; I) \hat{L}\hat{L}'\hat{l}\hat{l}' \frac{E(S\dot{S})}{(4\pi)^2(2J_i+1)} \right\} f^2(q^2) \rho_N(E_f). \tag{49}$$

Using this expression, we may hope to determine the amplitudes $A(S; I)$, which tell us how well the ground state is described by two clusters interacting with each other to form the state S.

6. Single-nucleon emission

As stated in the introduction, the main purpose of this paper is to try to give a better description of the quasi-elastic peak, which is largely due to single-nucleon emission. Equation (47) can be adapted for single-nucleon emission by letting $J_A = \dot{J}_A = J_A' = \frac{1}{2}$ and inserting values of J_i, J_B etc. appropriate for a particular nucleus. Then the 3–j, 6–j and 9–j symbols of equation (47) may be readily evaluated. Since fragment A now contains only one nucleon, there is only one internal coordinate in A, $x_1 = 0$. Hence

$$T_l^{(1)}(A; \tfrac{1}{2}\tfrac{1}{2}) = \delta_{l,0}\sqrt{2} \frac{q_\mu^2}{q^2} \left\{ e_1 + \frac{q^2}{8m^2}(e_1 - 2\mu_1) \right\} \tag{50}$$

from equation (37). Matrix elements of the type $T_l^{(1)}(B; J_B' J_B)$ may be evaluated in a

harmonic oscillator model, for example, and the matrix elements $T_L^{(1)}(a; J'J)$ may be evaluated using, say, a harmonic oscillator interaction between the fragments in the initial state and a square-well interaction between them in the final state. However, as found in I, matrix elements of this type may be approximated by using a plane-wave final state, and adjusting the relative momentum of the fragments to simulate the effects of an interaction potential. This procedure simplifies the numerical work considerably, and will be adopted here. To carry out this approximate calculation, it is easier to return to the operator H' given in equation (16) than to use equation (47).

If cluster A contains only one particle, then in equation (15)

$$x_1 = 0, \qquad \pi_1 = 0$$
$$r = r_1 - R_{A-1}.$$

We may combine the longitudinal part of the current operator of equation (16) with the scalar term as in equation (37), or directly using the conservation of nuclear current (de Forest and Walecka 1966). Then from (15) H_A' becomes

$$H_A' = \frac{q_\mu^2}{q^2}\left\{e_1 + \frac{q^2}{8m^2}(e_1 - 2\mu_1)\right\} + \alpha \cdot \frac{i\mu_1}{2m}(q \times \sigma_1)$$

and from (16)

$$H' = \left[\exp\left(iq \cdot r\frac{\mu}{M_A}\right)H_A' - \frac{1}{2M_A}\alpha \cdot \left\{p \exp\left(iq \cdot r\frac{\mu}{M_A}\right) + \exp\left(iq \cdot r\frac{\mu}{M_A}\right)p\right\}_T\right]$$

$$+ \exp\left(-iq \cdot r\frac{\mu}{M_B}\right)H_B' - \left\{\sum_{\beta\in B} e_\beta \exp(iq \cdot x_\beta)\right\}$$

$$\times \frac{1}{2M_B}\alpha \cdot \left\{-p \exp\left(-iq \cdot r\frac{\mu}{M_B}\right) - \exp\left(-iq \cdot r\frac{\mu}{M_B}\right)p\right\}_T \tag{51}$$

where the transverse component v_T of a vector v is perpendicular to q:

$$v_T = v - \hat{q}(v \cdot \hat{q}).$$

H_B' can be written

$$H_B' = \sum_{j\in B}\left[\frac{q_\mu^2}{q^2}\left\{e_j + \frac{q^2}{8m^2}(e_j - 2\mu_j)\right\}\exp(iq \cdot x_j)\right.$$

$$- \alpha \cdot \frac{e_j}{2m}\left\{\pi_j \exp(iq \cdot x_j) + \exp(iq \cdot x_j)\pi_j\right\}_T\left(1 - \frac{1}{B}\right)$$

$$\left.+ \alpha \cdot \frac{i\mu_j}{2m}q \times \sigma_j \exp(iq \cdot x_j)\right]. \tag{52}$$

The first part of equation (51) in square brackets is exactly the same as the interaction used in I. The second part of (51) consists of the terms which were neglected in I.

We now change our notation slightly, and let A be the atomic number of the original nucleus. For brevity, we consider only the longitudinal part of the operator (51) which becomes

$$H_L' = \exp\left\{iq \cdot r\left(1 - \frac{1}{A}\right)\right\}\frac{q_\mu^2}{q^2}\left\{e_1 + \frac{q^2}{8m^2}(e_1 - 2\mu_1)\right\}$$

$$+ \exp\left(\frac{-iq \cdot r}{A}\right)\left[\sum_{j\in B}\frac{q_\mu^2}{q^2}\left\{e_j + \frac{q^2}{8m^2}(e_j - 2\mu_j)\right\}\exp(iq \cdot x_j)\right]. \tag{53}$$

Matrix elements of the first part of the operator may be found easily: they are the ones used in I. The second part of the operator is slightly more difficult to use, since it involves both r and the coordinates $x_j = r_j - R_{A-1}$. For the part of the operator in square brackets, we

use the prescription discussed in §4: we replace x_j by r_j, use the harmonic oscillator model to evaluate the matrix elements, and finally multiply by

$$\exp\left\{\frac{q^2}{4(A-1)k_c}\right\}$$

according to the result of Tassie and Barker (1958). Here, k_c is the oscillator constant for the 'nucleus' of $A-1$ particles.

Let $|i\rangle$ and $|f\rangle$ represent the initial and final states of the ejected particle. Then the matrix element of H_L' of equation (53) is

$$M = \left\langle f\left|\exp\left\{iq \cdot r\left(1-\frac{1}{A}\right)\right\}\frac{q_\mu^2}{q^2}\left\{e_1+\frac{q^2}{8m^2}(e_1-2\mu_1)\right\}\right|i\right\rangle$$

$$+ \left\langle f\left|\exp\left(\frac{-iq \cdot r}{A}\right)\right|i\right\rangle\sum_k\left\langle k\left|\frac{q_\mu^2}{q^2}\left\{e_k+\frac{q^2}{8m^2}(e_k-2\mu_k)\right\}\exp(iq \cdot r_k)\right|k\right\rangle$$

$$- \sum_k\left\langle f\left|\exp\left(\frac{-iq \cdot r}{A}\right)\right|k\right\rangle\left\langle k\left|\frac{q_\mu^2}{q^2}\left\{e_k+\frac{q^2}{8m^2}(e_k-2\mu_k)\exp(iq \cdot r_k)\right\}\right|i\right\rangle. \qquad (54)$$

The sum Σ_k is over all occupied single-particle states of the residual nucleus, which are assumed orthogonal to $|i\rangle$, as is usual in a model such as the one we are using (Do Dang *et al.* 1968). The last term comes from antisymmetrization, and we have supposed that the residual nucleus does not change its state in the reaction.

Calculations have been carried out for ^{12}C, which was also discussed in I. For example, when an s-shell proton is emitted, the matrix element of (53) is

$$M = \sqrt{8}\left(\frac{\pi}{k}\right)^{3/4}\left(\exp\left[-\frac{\{p-q(1-1/A)\}^2}{2k}\right]+\exp\left\{\frac{-(p+q/A)^2}{2k}\right\}\right.$$

$$\left.\times \exp\left(-\frac{q^2}{4k_c}\frac{A-2}{A-1}\right)\left\{5-\frac{2q^2}{3k_c}-\frac{2q^2}{3A(kk_c)^{1/2}}-\frac{2}{3(kk_c)^{1/2}}p \cdot q\right\}\right).$$

The oscillator constant for the wave function of the relative coordinate is k. To obtain the cross section, we must take the square of the modulus of this matrix element and integrate over directions of p. As in I, the calculations were compared with the experimental results of Bounin and Bishop (1961). In this experiment, electrons of energy 194 MeV were scattered at 135° from ^{12}C. Under these conditions the momentum of the emitted proton increases from 0 at the threshold for disintegration to about 2 fm^{-1} at the high-energy tail. Since q is approximately constant over the whole peak, with value 1·4, $q/12 \sim 0·1$, and so $\exp\{-(p+q/A)^2/2k\} \sim \exp(-p^2/2k)$ over a large part of the quasi-elastic peak, and $\exp(-q^2/4k) \sim 0·18$. Hence the contribution to the matrix element from the indirect processes is largest near the low-energy transfer end of the quasi-elastic peak, and becomes smaller as the energy transferred increases. The same comments apply to the other matrix elements which are required to give the complete cross section. These features are clearly seen in the results which are presented below.

7. Results and conclusions

In figure 1 the cross section for single-nucleon emission is shown on the assumption that there is no interaction between the emitted particle and the residual nucleus. Curve B was obtained by using the Hamiltonian of equation (51), while curve A was obtained by keeping only the direct terms of (51), and is the same as curve A of figure 3 of I. It can be seen that the cross section is in fact decreased by using the complete Hamiltonian. The magnitude of the effect is appreciable, though not nearly large enough to give agreement with experiment. Recalling that the peak in the cross section occurs when $p \simeq q$, we see that the cross section is decreased for $p < q$, and practically unaltered for $p > q$. The

actual maximum value of the cross section at $p \simeq q$ is only decreased by about 3 %, although the correction is more important for $p < q$. The position of the maximum in B is slightly different from that in A, but only by about 2 MeV.

The actual magnitude of these corrections is determined to a large extent by the factor $\exp[-(q^2/4k_c)\{(A-2)/(A-1)\}] \sim 0.18$, the effect of which is largest when q and A are small. Transitions in which the residual nucleus changes its state have not been considered at all, but since they lead to different final states, they cannot interfere with the process considered here, and so will increase the cross section, making agreement with experiment even worse. Since the cross section for these processes will contain the factor $(\exp[-(q^2/4k_c) \times \{(A-2)/(A-1)\}])^2 \simeq 0.03$, they are expected to contribute very little to the cross section.
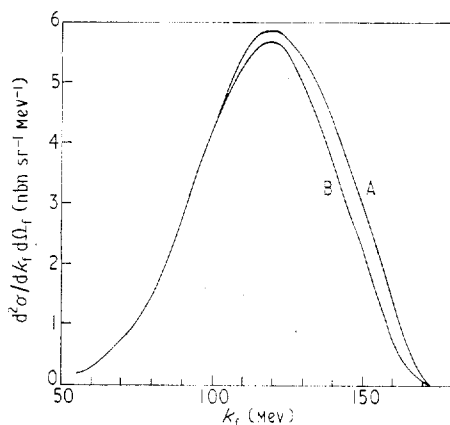

Figure 1. The quasi-elastic peak for ^{12}C calculated as described in the text. The units of cross section are 10^{-33} cm^2 sr^{-1} MeV^{-1}.

As discussed in I, the introduction of an effective potential to describe the interaction between the emitted nucleon and the residual nucleus hardly alters the shape of the spectrum but moves it bodily to the region of lower energy transfer. Calculations have been performed for several potentials, and the corrections which result from using the complete Hamiltonian are very similar to those shown in figure 1. They will therefore not be presented here.

The calculations on single-nucleon emission show that the indirect terms give small, though not negligible, corrections to the cross section. This implies that in the emission of larger fragments, the corrections will be important for light nuclei and small values of q.

Agreement with experiment (Bounin and Bishop 1961) is improved slightly by introducing indirect interactions, but it is still not very good. The results of Czyż (1963) are much better than those presented here. The reason for this is that, in the Fermi gas model with harmonic oscillator momentum distribution used by Czyż, there is a fairly large probability that the high-momentum states into which it is kinematically possible for the nucleons to be emitted are already filled. Since this would violate the Pauli exclusion principle, the cross section is diminished, and by a considerable amount. Under the experimental conditions considered above, the cross section is reduced to about half by the Pauli principle, and the free nucleons have energies of about 50 MeV at the peak. This seems rather unreasonable, and there is probably some other reason for the cross section being smaller than is calculated above. Recently, Ciofi degli Atti (1968) has found that short-range dynamical correlations can have a large effect on the elastic electron scattering form factor, even at momentum transfers as low as 1.5 fm^{-1}. This being the case, the quasi-elastic scattering cross section may be considerably diminished by short-range correlations. Preliminary calculations indicate that this is indeed the case, and a full description will be published when more results are available.

Acknowledgments

Appendix

The following summations are necessary to derive equation (33):

(i) $$\sum_{\mu} Y^*_{\lambda\lambda'\mu} \cdot Y_{\lambda\lambda''\mu} = \delta_{\lambda'\lambda''} \frac{2\lambda+1}{4\pi}$$

(ii) $$\sum_{\mu} \boldsymbol{u} \cdot Y^*_{\lambda\lambda'\mu} \, \boldsymbol{v} \cdot Y_{\lambda\lambda''\mu} = \boldsymbol{u} \cdot \boldsymbol{v} \frac{1}{3} \frac{2\lambda+1}{4\pi} \delta_{\lambda'\lambda''}$$
$$+ (-)^{\lambda+1} \frac{1}{\sqrt{2}} \frac{2\lambda+1}{4\pi} i \begin{Bmatrix} 1 & 1 & 1 \\ \lambda & \lambda' & \lambda'' \end{Bmatrix} \hat{\lambda}'\hat{\lambda}'' \sqrt{3} \begin{pmatrix} \lambda' & \lambda'' & 1 \\ 0 & 0 & 0 \end{pmatrix} (\boldsymbol{u} \times \boldsymbol{v}) \cdot \hat{\boldsymbol{q}}$$
$$+ (-)^{\lambda} \sqrt{\frac{15}{2}} \frac{2\lambda+1}{4\pi} \begin{Bmatrix} 2 & 1 & 1 \\ \lambda & \lambda' & \lambda'' \end{Bmatrix} \hat{\lambda}'\hat{\lambda}'' \begin{pmatrix} \lambda' & \lambda'' & 2 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{q^2} \{ (\boldsymbol{q}\cdot\boldsymbol{u})(\boldsymbol{q}\cdot\boldsymbol{v}) - \tfrac{1}{3}q^2\boldsymbol{u}\cdot\boldsymbol{v} \}$$

(iii) $$\sum_{\mu} \boldsymbol{q} \cdot Y^*_{\lambda\lambda'\mu} \times Y_{\lambda\lambda''\mu} = q(-)^{\lambda} \begin{Bmatrix} 1 & 1 & 1 \\ \lambda & \lambda' & \lambda'' \end{Bmatrix} \frac{2\lambda+1}{4\pi} \sqrt{6}\, i \hat{\lambda}'\hat{\lambda}'' \begin{pmatrix} \lambda' & \lambda'' & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

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